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A sufficient condition for w-Hyperbolic convolution operators

by

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1. Introduction

Different types of hyperbolicity for differential and convolution operators has been studied by many authors, for example see [2], [3] and [4].

In this paper we will study hyperbolicity in the following sense.

Let ε_w be a Berling distribution with compact support. $S \in \varepsilon_w$ is hyperbolic with respect to a real number N if S has a fundamental solution E in convolution sense (that is E is a Berling distribution and $S*E=\delta$) where δ is the Dirac measure) such that the support of E is contained in the set $\{x: x \in R$ and $xN \geqslant 0\}$ wnere R is the set of all real numbers.

2. Berling distributinos

In this section we will recall the defintion of Berling distaibutions and state a theorem which we will use here. For more details you may see [1].

Let M denotes the class of all real valued functions w, defined on R such that the following four conditions hold:

(I) $0=\lim_{x\to 0} w(x)=w(0)\leqslant w(x+y)\leqslant w(x)+w(y)$ for all real numbers x and y.

(II) $\int_{-\infty}^{\infty} w(x)/(1+|x|)^2 dx<\infty$, where $|x|$ is the absolute value of x.

(III) For some real number a and positive number b, $w(x)\geqslant a+b\log(1+|x|)$ for every real number x.

(IV) dw/dx exists and bounded on R.

DEFINITION. Given $w \in M$ and a positive integer n. Let $D_w([-n, n])$ be the vector space of Lebesgue integrable functions ϕ in R with support in the compact interval $[-n, n]$, such that for all $r\geqslant 0$,

$$||\phi||_r^{(w)}=\int_{-\infty}^{\infty}|\hat{\phi}(x)|e^{rw(x)}dx$$

where $\hat{\phi}$ is the Fourier-Laplace transform of the function ϕ which is given by the formula

$$\hat{\phi}(x)=\int_{-\infty}^{\infty}\phi(y)e^{-iyx}dx$$

Let $D_w=D_w(R)=\bigcup_{n=1}^{\infty}D_w([-n, n])$. It is clear that D_w is a vector space, with respect to the usual definitions of addition and scalar multiplication.

The Berling distribution D'_w is the space of all continuous linear functional on D_w.

An equivalent definition is:

D'_w is the speace of all linear functionals u on D_w such that for each compact enterval $[-n, n]$ there exist $r\geqslant 0$ and a constant C_1 such that

$$|u(\phi)|\leqslant C_1||\phi||_r^{(w)}$$

For every ϕ in $D_w([-n, n])$ D'_w is given the weak topology, that is the topology given by the seminorms

$$u\to|u(\phi)|\,,\quad \text{where } \phi \text{ is any element of } D_w$$

ε_w is the set of all functions ϕ on R such that if $f\in D_w$ then $f\phi\in D_w$.

ε_w is the space of all continuous linear functional on ε_w.

THEOREM 1. (Paley-Wiener theorem for test functions) *An entire function $U(z)$ is the Fourier-Laplace transform of a function ϕ in D_w if and only if for every $r>0$ and $m>0$ there is a constant A such that*

$$|U(z)|\leqslant Ae^{-rw(x)+m|y|+H(y)}$$

where $z=x+iy$, x and y are element in R, and $H(y)=\sup\{ay\colon a\in$ support of $\phi\}$.

For the proof of this theorem, see [1] Theorem 1.4.1. page 365.

3. A sufficient condition for w-hyperbolicity

THEOREM. *Let $S\in\varepsilon_w$. If $\hat{S}$, the Fourier-Laplace transform of S, satisfying the following condition:*

For every real number h there is a constant F such that

$$(1)\qquad |\hat{S}(x+itN)|\geqslant e^{ht}\,,\quad \text{where}\quad t\leqslant -F(1+w(x))\,,\quad x\in R\,.$$

Then S is w-hyperbolic with respect to N.

Proof. Let $\hat{S}$ satisfies (1), since $t\leq -F(1+w(x))$ then there is a constant $q\leqslant -F$ such that

$$t=t(x)=q(1+w(x))\,.$$

Let L_q be a set of complex numbers defined by

$$L_q=\{z\colon z\cdot x+itN,\ t=t(x)=q(1+w(x))\quad\text{and}\quad i=\sqrt{-1}\}$$

Let $z\in L_q$ then by using (1) weh have

$$|\hat{S}(z)| \geqslant e^{ht} \tag{2}$$

Let $\phi \in D_w$ and define,

$$\check{E}(\phi) = \frac{1}{2\pi}\int_{L_q} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz \tag{3}$$

where $\check{E}(\phi) = E(\check{\phi})$ and $\check{\phi}(x) = \phi(-x)$.

We will prove that $E \in D'_w$.

Consider $\phi_1(V) = \phi(V)e^{tNV}$ then

$$\hat{\phi}(z) = \hat{\phi}(x+itN) = \int_{-\infty}^{\infty} \phi(V) e^{-iv(x+itN)} dv$$
$$= \int_{-\infty}^{\infty} \phi_1(V) e^{-ivx} dx = \hat{\phi}_1(x) .$$

Take $T = Ee^{tNV}$ then

$$\check{T}(\phi_1) = T(\check{\phi}_1) = T(\check{\phi}e^{-tNV}) = e^{-tNV}T(\check{\phi})$$
$$= E(\check{\phi}) = \check{E}(\phi) = \frac{1}{2\pi}\int_{L_q} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz$$
$$= \frac{1}{2\pi}\int_{L_q} \frac{\hat{\phi}_1(x)}{\hat{S}(z)} dz$$

But $z = x + itN = x + iqN(1+w(x))$, so

$$\frac{1}{2\pi}\int_{L_q} \frac{\hat{\phi}_1(x)}{\hat{S}(z)} dz = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{\phi}_1(x)}{\hat{S}(z(x))}\left(1 + iqN\frac{dw}{dx}\right) dx$$

But $w \in M$, so by condition (IV) there is a constant C such that

$$\left|\frac{dw}{dx}\right| \leqslant C \quad \text{for every} \quad x \in R \tag{4}$$

Then,

$$|\check{T}(\phi_1)| \leqslant \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} \frac{\hat{\phi}_1(x)}{\hat{S}(Z(x))}\left(1 + iqN\frac{dw}{dx}\right) dx\right.$$
$$= \frac{1}{2\pi}(1 + q^2N^2C^2)^{1/2}\int_{-\infty}^{\infty} \frac{|\hat{\phi}_1(x)|}{|\hat{S}(z(x))|} dx$$

Making use of (2) we get

$$|\check{T}(\phi_1)| \leqslant \frac{1}{2\pi}(1 + q^2N^2C^2)^{1/2}\int_{-\infty}^{\infty} e^{ht} |\hat{\phi}_1(x)| dx$$

But $t = q(1 + w(x))$ then

$$|\check{T}(\phi_1)| \leqslant C_1 \int_{-\infty}^{\infty} e^{rw(x)} |\hat{\phi}(x)| dx = C_1 \|\phi_1\|_r^{(x)}$$

where $r = hq$ and $C_1 = (1/2\pi)e^{r(1+q^2N^2C^2)1/2}$.

This implies that $\check{T} \in D_w$ and so $E \in D'_w$.

Next we want to show that E is a fundamental solution for S that is

$$S * E = \delta \, .$$

Let $\phi \in D_w$ then

$$(S*E)^{v}(\phi) = (S*E)(\check{\phi}) = E(S*\phi)^{v} = \check{E}(S*\phi)$$

So by using (3) we have

$$(S*E)^{v}(\phi) = \frac{1}{2\pi}\int_{L_q} \frac{\hat{S}(z)\hat{\phi}(z)}{\hat{S}(z)} dz = \frac{1}{2\pi}\int_{L_q} \hat{\phi}(z) dz$$

By Theorem (1.) $\hat{\phi}(z)$ is an entire function and so by using Cauchy theorem we obtain

$$\int_{L_q} \hat{\phi}(z) dz = \lim_{r \to \infty} \int_{-r}^{r} \phi(x + i t_r N) dx$$

where $t_r = q(1 + w(r))$.
Also,

$$\begin{aligned} \hat{\phi}(x + i t_r N) &= \int_{-\infty}^{\infty} \phi(v) e^{-iv((x + i t_r)N)} dv \\ &= \int_{-\infty}^{\infty} \phi_1(v) e^{-ivx} dx = \hat{\phi}_1(x) \end{aligned}$$

where $\phi_1(v) = \phi(v) e^{v t_r N}$. Hence

$$(S*E)^{v}(\phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_1(x) dx = \phi_1(0) = \phi(0)$$

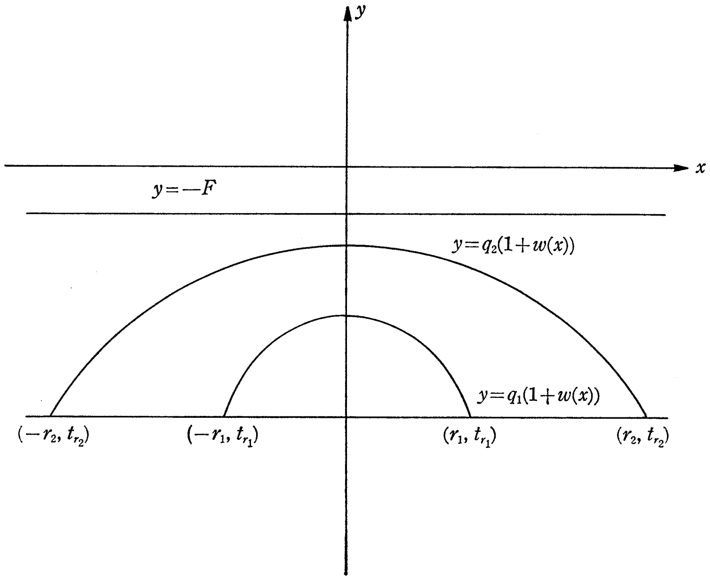
Therefore $(S*E)^{v} = \delta$ or $S*E = \delta$.
Recall that (3) says that

$$\check{E}(\phi) = \int_{L_q} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz$$

We claim that E is independent of q as long as $q \leqslant -F$. Since $S \in \varepsilon_w$ then by Theorem 1.8.14 in [1] page 379, $\hat{S}$ is an entire function. Also $\hat{\phi}$ is an entire function by Theorem 1.
So the function $\hat{\phi}(z)/\hat{S}(z)$ is analytic in L_q.
Now making use of Cauchy theorem we have

$$\int_{L_q} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz = \lim_{r_1 \to \infty} \int_{-r_1}^{r_1} \frac{\hat{\phi}(x + i t_{r_1} N)}{\hat{S}(x + i t_{r_1} N)} dx$$

where $t_{r_1} = q_1(1 + w(r_1))$ and $q_1 \leqslant -F$



and

$$\int_{L_{q_2}} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz = \lim_{r_2 \to \infty} \int_{-r_2}^{r_2} \frac{\hat{\phi}(x + it_{r_2}N)}{\hat{S}(x + it_{r_2}N)} dx$$

where $t_{r_2} = q_2(1 + w(r_2))$ and $q_2 \leqslant -F$. But as we see from the figure $t_{r_1} = t_{r_2}$, so as $r_1 \to \infty$ and $r_2 \to \infty$ we have

$$\int_{L_{q_1}} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz = \int_{L_{q_2}} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz .$$

It remains to show that the support of E is contained in the set

$$\{x \colon x \in R \quad \text{and} \quad xN \geqslant 0\} .$$

So, given $\varepsilon > 0$ and $\phi \in D_w$ with support contained in the set

$$\{x \colon x \in R \quad \text{and} \quad xN > \varepsilon\} .$$

Making use of (1) and Theorem (1.) we have:

If $z \in L_q$ then for every $r > 0$ and $m > 0$ there is a constant A such that

$$\left| \frac{\hat{\phi}(z)}{\hat{S}(z)} \right| \leqslant A e^{-rw(x) + m|tN| + H(tN) - ht}$$

where $t = t(x) = q(1 + w(x))$, $q \leqslant -F$ and

$$\begin{aligned} H(tN) &= \sup\{xtN \colon x \in \text{support of } \phi\} \\ &= \sup\{xq(1 + w(x))N \colon xN > \varepsilon\} \end{aligned}$$

But we will take $q < 0$, so noting $\varepsilon > 0$ and $w \geqslant 0$ to get

$$H(tN) \leqslant \varepsilon q' .$$

Hence,

$$\left|\frac{\hat{\phi}(z)}{\hat{S}(z)}\right| \leqslant Ae^{-rw(x)-qm(1+w(x))|N|+\varepsilon q-hq(1+w(x))}$$

choose $\varepsilon > h+m|N|+1$, then

$$\frac{|\hat{\phi}(z)|}{|\hat{S}(z)|} \leqslant Ae^{-(r+mNq+hq)w(x)+q}$$

But $w \in M$, so by condition (III) we have

$$w(x) \geqslant a+b\log(1+|x|) \text{ for some } a \text{ and positive } b .$$

Choose $r > -hq-m|N|q+2/b$ then

$$\left|\frac{\hat{\phi}(z)}{\hat{S}(z)}\right| \leqslant \frac{Be^{q}}{(1+|x|)^2} \tag{5}$$

where $B = Ae^{-a}$.

Now,

$$\begin{aligned} E^{v}(\phi) &= \int_{L_q} \frac{\hat{\phi}(z)}{\hat{S}(z)} dz \\ &= \int_{-\infty}^{\infty} \frac{\hat{\phi}(z(x))}{\hat{S}(z(x))} \left(1+iqN\frac{dw}{dx}\right) dx . \end{aligned}$$

Use (4) and (5) to get

$$|\check{E}(\phi)| \leqslant |(1+N^2C^2q^2)^{1/2}Be^{q} \int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2}$$

So,

$$|\check{E}(\phi)| \leqslant Q(1+N^2C^2q^2)^{1/2}e^{q}$$

where

$$Q = B\int_{-\infty}^{\infty} \frac{dx}{(1+|x|)^2}$$

But E is independent on q as long as $q \leqslant -F$, so as $q \to -\infty$ we have

$$\check{E}(\phi) = 0$$

which implies that support of $E \subset \{x : x'N > \varepsilon\} \subset \{x : xN \geqslant 0\}$ as required, so the proof is completed.

References

[1] BJORCK, G.; Linear Partial Differential Operators and Generalized Distributions, *Arkiv for Matematic*, (1966) 351–407.

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[4] LARSSON, E.; Generalized Hyperbolicity, *Arkiv for Matematik*, *Band* 7 *No*. 2, 1967, 11–32.

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